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ON THE MINIMAL DIMENSION OF THE AMBIENT SPACE OF A PROJECTIVE SCHEME

by

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In [5], Lluis has shown that if X is a projective variety of dimension n over the infinite field k, then X may be embedded as a (closed) subvariety of $\mathbb{P}_k^m$, where

$$m = \max\{2n+1, d+n-1\},$$

$d = \max\{\dim(\underline{m}_{X,x}/\underline{m}_{X,x}^2) \mid x \in X\}$ and $\underline{m}_{X,x}$ denotes the maximal ideal of the local ring $O_{X,x}$ of X at x.

As a corollary of the formal theory in [2] we have shown that the conclusion of this theorem holds for any projective scheme X such that the dimension of the singular locus of X is less than the dimension of X. In the general case, one gets

$$m = \max\{2n+1, d+n\}$$

where n and d are as before.

On the other hand, the techniques applied by Lluis might be usefull in dealing with "embedding theorems" for projective morphisms (see [3]): Given a family of projective schemes parametrized by some scheme Y, find a uniform embedding of this family. In other words, if $g: X \to Y$ is a projective morphism (with Y quasi-compact), find minimal N such that

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "g"'] & \mathbb{P}_Y^N \arrow[d] \\ & Y \end{tikzcd}$$

commutes.

In this note we extend Lluis' techniques to any projective scheme over an infinite field k. For notations we refer to

[1] and [4].

So let X be a closed subscheme of $\mathbb{P}^N_k$, where k is an infinite field, and let $i\colon X \hookrightarrow \mathbb{P}^N_k$ be the closed embedding.

Let $\Omega^1_{X/k}$ and $\Omega^1_{\mathbb{P}^N/k}$ denote the Modules of differentials on X and $\mathbb{P}^N_k$, respectively. The canonical surjective

$$\Omega^1_{\mathbb{P}^N_k/k} \longrightarrow i_*(\Omega^1_{X/k})$$

gives a closed embedding j such that the following diagram commutes:

(1)
$$\begin{tikzcd} \mathbb{V}(\Omega^1_{\mathbb{P}^N_k/k}) \arrow[dr, "p"'] & & \mathbb{V}(i_*(\Omega^1_{X/k})) \arrow[ll, hook', "j"'] \arrow[dl, "q"] \\ & \mathbb{P}^N_k = \mathrm{Proj}(k[Z_0,\dots,Z_N]) & \end{tikzcd}$$

where p and q are the canonical morphisms.

If $U = D_+(Z_0)$, we get $p^{-1}(U) = \mathbb{V}(\Omega^1_{D_+(Z_0)/k}) = D_+(Z_0) \times \mathbb{A}^N_k$, the identification being done by

$$\Omega^1_{D_+(Z_0)/k} = k\left[\frac{Z}{Z_0}\right] d\frac{Z_1}{Z_0} \oplus \dots \oplus k\left[\frac{Z}{Z_0}\right] d\frac{Z_N}{Z_0}$$

Let Y be a closed subscheme of X. Put

$$Z_0'(X,Y) = \mathrm{pr}_2(\mathbb{V}(i_*(\Omega^1_{X/k})) \cap (Y \cap D_+(Z_0)) \times \mathbb{A}^N_k)$$

and finally identifying $\mathbb{A}^N_k$ with $D_+(Z_0)$ in $\mathbb{P}^N_k = \mathrm{Proj}(k[Z_0,\dots,Z_N])$, we let $Z_0(X,Y)$ denote the closure of $Z_0'(X,Y)$ in $\mathbb{P}^N_k$. Similarly one defines $Z_i(X,Y)$ by Z_i instead of Z_0.

<u>Lemma 1</u>. Assume $k = \bar{k}$. Then $Z_i(X,Y)$ contains the Zariski tangent space of X at all closed points $x \in Y \cap D_+(Z_i)$.

<u>Proof</u>. We may assume that $i = 0$.

Taking the fibers over the point x , (1) induces

$$\begin{array}{ccc} \mathbb{V}(\Omega^1_{\mathbb{P}^1_k/k})_x & \xhookleftarrow{\;j_x\;} & \mathbb{V}(i_*(\Omega^1_{X/k}))_x \\ \varphi \big\downarrow \wr & & \\ \mathbb{A}^N_k & & \end{array}$$

where the isomorphism φ , which is induced by pr_2 , is nothing but the canonical isomorphism between $\mathbb{A}^N_k$ and the Zariski tangent space of $\mathbb{P}^N_k$ at x . Hence $j_x\varphi$ maps $V(i_*(\Omega^1_{X/k}))_x$ isomorphically onto the Zariski tangent space of X at x .

Q.E.D.

<u>Lemma 2</u>. $\dim(Z_i(X,Y)) \leq \dim(Y) + \max\{rk_{k(x)}(\underline{m}_{X,x}/\underline{m}_{X,x}^{\,2}) \mid x \in Y\} = z_i(X,Y)$

<u>Proof</u>. It suffices to show this for $Z_i'(X,Y)$. For this, it's enough to show that

$$\dim(\mathbb{V}(i_*(\Omega^1{}_{X/k})) \cap (Y \cap D_+(Z_i)) \times \mathbb{A}^N_k)) \leq z_i(X,Y) .$$

Let $y \in Y \cap D_+(Z_i)$. It suffices to show that

$$\dim[\mathbb{V}(i_*(\Omega^1_{X/k})) \cap (Y \cap D_+(Z_i) \times \mathbb{A}^N_k))^-_y] \leq$$

$$\max\{rk_{k(x)}(\underline{m}_{X,x}/\underline{m}_{X,x}^{\,2}) \mid x \in Y\} ,$$

but this is clear since

$$\mathbb{V}(i_*(\Omega^1_{X/k})) \cap (Y \cap D_+(Z_i) \times \mathbb{A}^N_k))_y$$

$$\cong \mathbb{V}(\Omega^1_{X/k}(y)) \cong \mathbb{V}(\underline{m}_{X,y} / m_{X,y}{}^2) \ .$$

Q.E.D.

Now let π_x denote the blowing up with center in the k-point $x \in \mathbb{P}^N_k$. We get (cf. [3]) the diagram

$$\begin{tikzcd}
\widetilde{\mathbb{P}^N_k} \arrow[r, hook] \arrow[d, "\pi_x"'] \arrow[dr, "\lambda_x"] & \mathbb{P}^N_k \times \mathbb{P}^{N-1}_k \arrow[dl, "pr_1"'] \arrow[d, dotted] \\
\mathbb{P}^n_k & \mathbb{P}^{N-1}_k
\end{tikzcd}$$

λ_x induces a morphism $\overline{\lambda}_x : \mathbb{P}^N_k - \{x\} \to \mathbb{P}^{N-1}_k$, the projection with center $\{x\}$. If $x = \{1:0:\ldots:0\}$, we get $\lambda : \mathbb{A}^N_k - \{(0,\ldots,0)\} = D_+(Z_0) - \{(0,\ldots,0)\} \to \mathbb{P}^{N-1}_k$, which is nothing but the canonical $(\alpha_0,\ldots,\alpha_N) \longmapsto (\alpha_0:\ldots:\alpha_N)$.

There exists a closed subscheme $Sc(X)$ of $\mathbb{P}^N_k$, which contains all lines in $\mathbb{P}^N_k$ with 2 or more points in common with X and (hence, cf. [3]) the tangential cone of X at x for all $x \in X$. In particular $Sc(X)$ contains the Zariski tangent space of X at all smooth points $x \in X$. Moreover, $\dim Sc(X) \leq 2\dim(X) + 1$, cf. [3].

<u>Theorem</u>. Let k be an infinite field. A projective scheme X over k may be embedded as a closed subscheme of $\mathbb{P}^m_k$, where

$$m = \max\{2\dim(X) + 1 \ , \ d + \dim(S(X))\}$$

where $S(X)$ denotes the (closed) subset of X consisting of the non-smooth points.

Proof. X is a closed subscheme of some $\mathbb{P}_k^N$. Assume $N > m$. Since k is infinite, there is a k-point P not contained in

$$Se(X) \cup Z_0(X,S(X)) \cup \ldots \cup Z_N(X.S(X)) .$$

λ_P induces a morphism

$$\varphi : : X \to \lambda_P(X) = \overline{X} \hookrightarrow \mathbb{P}_k^{N-1}$$

(where $\lambda_P(X)$ denotes the scheme-theoretic image, cf. [1] (I. 9.5)).

To show is that φ is an isomorphism. For this it suffices to show that $\varphi \otimes_k \overline{k}$ is an isomorphism, cf. [1], (VI,2.7.1). Hence we may assume $k = \overline{k}$.

Moreover, we may assume that $P = (1:0:\ldots:0)$.

Since $P \notin Sc(X)$, it follows that φ is bijective on closed points, and hence bijective on the underlying topological spaces. Since, furthermore, φ is proper, it's a homeomorphism.

Moreover, φ is unramified: Indeed, for this it suffices to show that φ is unramified at all closed points, i.e., that if $x \in X$ is a closed point, then

$$\underline{m}_{\overline{X},\varphi(x)} O_{X,x} = \underline{m}_{X,x} ,$$

i.e., that the canonical

$$\underline{m}_{\overline{X},\varphi(x)}/\underline{m}_{\overline{X},\varphi(x)}^2 \to \underline{m}_{X,x}/\underline{m}_{X,x}^2$$

is surjective. But this is clear, since P is choosen outside the Zariski tangent spaces of X at all closed points of X , so that the induced morphism from the tangent space of X at x to the tangent space of $\overline{X}$ at $\varphi(x)$ is a closed embedding.

Finally, the fact that φ is bijective immediately implies that $O_{X,x}$ is a finite $O_{\overline{X},\varphi(x)}$ - module for all closed points $x \in X$. Hence Nakayamas lemma gives that the canonical $O_{\overline{X},\varphi(x)} \to O_{X,x}$ is an isomorphism, and we are done.

Q.E.D.

References

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